

DICHOTOMY OF GLOBAL CAPACITY DENSITY IN METRIC MEASURE SPACES

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ABSTRACT. The variational capacity cap_p in Euclidean spaces is known to enjoy the density dichotomy at large scales, namely that for every $E \subset \mathbb{R}^n$,

$$\inf_{x \in \mathbb{R}^n} \frac{\text{cap}_p(E \cap B(x, r), B(x, 2r))}{\text{cap}_p(B(x, r), B(x, 2r))}$$

is either zero or tends to 1 as $r \rightarrow \infty$. We prove that this property still holds in unbounded complete geodesic metric spaces equipped with a doubling measure supporting a p -Poincaré inequality, but that it can fail in nongeodesic metric spaces and also for the Sobolev capacity in \mathbb{R}^n .

It turns out that the shape of balls impacts the validity of the density dichotomy. Even in more general metric spaces, we construct families of sets, such as John domains, for which the density dichotomy holds. Our arguments include an exact formula for the variational capacity of superlevel sets for capacitary potentials and a quantitative approximation from inside of the variational capacity.

1. INTRODUCTION

In extending a result of Hayman and Pommerenke [HP78] and giving a characterization of analytic functions mapping the unit disk into a given planar domain Ω , Stegenga [St80] came across a dichotomy property of the logarithmic capacity, namely that if $E \subset \mathbb{R}^2$ is the complement of a planar domain, then its logarithmic capacity density with respect to a radius $r > 0$ either tends to 0 or to 1 as $r \rightarrow \infty$. The property that the complement of Ω has its logarithmic capacity density tending to 1 at global scales characterizes the property that analytic functions from the unit disk to Ω belong to the class BMOA.

In [AI15] the first author, together with Itoh, studied such a dichotomy property of the global capacity density for the variational p -capacity, $1 < p < \infty$, in weighted Euclidean spaces. In this note we investigate the same problem in the nonsmooth setting of metric measure spaces, where it is considerably more complicated and subtle. It turns out that the dichotomy fails in general, and that the shape of balls plays a significant role.

Fix $1 < p < \infty$ and let (X, d, μ) be an unbounded complete metric measure space with a doubling measure μ supporting a p -Poincaré inequality. It is known that such a metric space is L -quasiconvex for some $L \geq 1$, i.e., for all $x, y \in X$, there exists a rectifiable

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curve γ connecting x and y with length $\ell(\gamma) \leq Ld(x, y)$. (See Section 2 for this and other facts mentioned in this introduction.) Define the *inner metric* d_{in} by

$$(1.1) \quad d_{\text{in}}(x, y) = \inf_{\gamma} \ell(\gamma),$$

where the infimum is taken over all rectifiable curves γ connecting x and y . It follows from the L -quasiconvexity that $d(x, y) \leq d_{\text{in}}(x, y) \leq Ld(x, y)$. Moreover, arc length with respect to the given distance d and with respect to the inner metric d_{in} are the same, and thus X is a geodesic space (i.e., 1-quasiconvex) with respect to d_{in} .

Now let $E \subset X$ and $\tau > 1$. We study the following global lower capacity densities

$$\begin{aligned} \mathcal{D}(r, \tau, E) &= \inf_{x \in X} \frac{\text{cap}_p(E \cap B(x, r), B(x, \tau r))}{\text{cap}_p(B(x, r), B(x, \tau r))}, \\ \mathcal{D}_{\text{in}}(r, \tau, E) &= \inf_{x \in X} \frac{\text{cap}_p(E \cap B_{\text{in}}(x, r), B_{\text{in}}(x, \tau r))}{\text{cap}_p(B_{\text{in}}(x, r), B_{\text{in}}(x, \tau r))}. \end{aligned}$$

Here $B(x, r) = \{y \in X : d(x, y) < r\}$ and $B_{\text{in}}(x, r) = \{y \in X : d_{\text{in}}(x, y) < r\}$ denote the ordinary and inner balls, respectively, and cap_p is the variational capacity (see (2.1)).

It is easy to see that, as $r \rightarrow \infty$, the limit of $\mathcal{D}(r, \tau, E)$ and that of $\mathcal{D}_{\text{in}}(r, \tau, E)$ are comparable (see Lemma 3.2). However, they have different nature. We show that $\mathcal{D}_{\text{in}}(r, \tau, E)$ has the same dichotomy as in the Euclidean case found in [AI15, Corollary 1.5], whereas $\mathcal{D}(r, \tau, E)$ does not have such a dichotomy in general. More precisely, we have the following two theorems.

Theorem 1.1. *For every $E \subset X$ one of the following statements holds:*

- (i) $\lim_{R \rightarrow \infty} \mathcal{D}_{\text{in}}(R, \tau, E) = 0$,
- (ii) $\lim_{R \rightarrow \infty} \mathcal{D}_{\text{in}}(R, \tau, E) = 1$.

Furthermore, the two possibilities listed above are independent of $\tau > 1$, and (i) holds if and only if any of the following equivalent conditions is satisfied:

- (a) $\lim_{R \rightarrow \infty} \mathcal{D}(R, \tau, E) = 0$,
- (b) $\mathcal{D}(r, \tau, E) = 0$ for all $r > 0$,
- (c) $\mathcal{D}_{\text{in}}(r, \tau, E) = 0$ for all $r > 0$.

Theorem 1.2. *There exists a complete unbounded metric measure space (X, d, μ) supporting a 1-Poincaré inequality with μ doubling and $E \subset X$ such that*

$$0 < \liminf_{R \rightarrow \infty} \mathcal{D}(R, \tau, E) < 1 \quad \text{for all } \tau > 1.$$

The above counterexample to the dichotomy arises from the lack of geodesics with respect to the ordinary metric. Although by the quasiconvexity of X , an ordinary ball $B(x, r)$ and an inner ball $B_{\text{in}}(x, r)$ satisfy

$$(1.2) \quad B_{\text{in}}(x, r) \subset B(x, r) \subset B_{\text{in}}(x, Lr),$$

and thus are comparable, the ordinary balls may be oddly shaped. This illustrates the difference between Theorems 1.1 and 1.2. As was observed in [AI15], uniform approximation of capacity from inside plays an important role for the dichotomy of the global capacity density. Such an approximation property can be verified for domains satisfying an interior corkscrew condition, see Section 6 for details. To further understand this phenomenon we introduce the notion of capacitarily stable collections of sets in Section 8 and

show that the dichotomy holds for such collections. We also give examples of capacitarly stable collections, including one consisting of John domains.

Even though there is no dichotomy of the type above for $\mathcal{D}(R, \tau, E)$, we have the following weak dichotomy.

Theorem 1.3. *Let $\tau > 1$. Then there is a constant $A > 0$, depending only on τ , p and X , such that for every $E \subset X$ one of the following statements holds:*

- (i) $\lim_{R \rightarrow \infty} \mathcal{D}(R, \tau, E) = 0$,
- (ii) $\liminf_{R \rightarrow \infty} \mathcal{D}(R, \tau, E) \geq A$.

Furthermore, the two possibilities listed above are independent of $\tau > 1$, with the exception that the constant A depends on τ .

One may ask if there can be a similar dichotomy for other capacities as well. In [AI16] the first author observed that the Riesz capacity of order α ($0 < \alpha \leq 2$) in the Euclidean space has the same dichotomy property. On the other hand, we show in Example 7.2 that the Sobolev capacity C_p has neither dichotomy nor weak dichotomy even in the linear case $p = 2$ on unweighted \mathbb{R}^n . It would be interesting to characterize capacities whose global densities have dichotomy.

The outline of the paper is as follows. In Section 2 we introduce the necessary background from nonlinear analysis on metric spaces. In Section 3 we recall some basic estimates for the variational capacity and use them to deduce comparison results for the capacity density functions \mathcal{D} and \mathcal{D}_{in} . In Section 4 we deduce an identity for the capacity of superlevel sets for the capacitary potentials. Similar estimates have earlier been obtained in [BMS01], but here we obtain an exact identity.

In the subsequent two sections, we give the proof of Theorem 1.1, through the use of a number of simpler lemmas. Also Theorem 1.3 is obtained therein. In Section 7 we give the key counterexample yielding Theorem 1.2, and another counterexample showing that there is no dichotomy for the Sobolev capacity.

Finally, in the last section we define capacitarly stable collections, show that they satisfy a dichotomy, and give examples of such families, including families of John domains and families of domains satisfying the interior corkscrew condition.

2. NOTATION AND PRELIMINARIES

We assume throughout the paper that $1 < p < \infty$ and that $X = (X, d, \mu)$ is an unbounded complete metric space equipped with a metric d and a doubling measure μ , i.e., there exists $C > 0$ such that for all balls $B = B(x_0, r) := \{x \in X : d(x, x_0) < r\}$ in X ,

$$0 < \mu(2B) \leq C\mu(B) < \infty.$$

Here and elsewhere we let $\lambda B = B(x_0, \lambda r)$. We will also assume that X supports a p -Poincaré inequality, see below, and that $\Omega \subset X$ is a nonempty bounded open set.

Proofs of the results in this section, as well as historical comments, can be found in the monographs [BB11] and [HKST15].

We will only consider curves which are nonconstant, compact and rectifiable (i.e., have finite length), and thus each curve can be parameterized by its arc length ds . A property is said to hold for p -almost every curve if it fails only for a curve family Γ with zero p -modulus, i.e., there exists $0 \leq \rho \in L^p(X)$ such that $\int_{\gamma} \rho ds = \infty$ for every curve $\gamma \in \Gamma$.

Following [KM98] and [HK98] we introduce weak upper gradients as follows.

Definition 2.1. A measurable function $g : X \rightarrow [0, \infty]$ is a p -weak upper gradient of a function $f : X \rightarrow [-\infty, \infty]$ if for p -almost every curve $\gamma : [0, \ell(\gamma)] \rightarrow X$,

$$|f(\gamma(0)) - f(\gamma(\ell(\gamma)))| \leq \int_{\gamma} g \, ds,$$

where the left-hand side is considered to be ∞ whenever at least one of the terms therein is infinite.

If f has a p -weak upper gradient in $L^p(X)$, then it has an a.e. unique *minimal p -weak upper gradient* $g_f \in L^p(X)$ in the sense that for every p -weak upper gradient $g \in L^p(X)$ of f we have $g_f \leq g$ a.e., see [S01]. Following [S00], we define a version of Sobolev spaces on the metric space X .

Definition 2.2. For a measurable function $f : X \rightarrow [-\infty, \infty]$, let

$$\|f\|_{N^{1,p}(X)} = \left(\int_X |f|^p \, d\mu + \inf_g \int_X g^p \, d\mu \right)^{1/p},$$

where the infimum is taken over all p -weak upper gradients g of f . The *Newtonian space* on X is

$$N^{1,p}(X) = \{f : \|f\|_{N^{1,p}(X)} < \infty\}.$$

The space $N^{1,p}(X)/\sim$, where $f \sim h$ if and only if $\|f - h\|_{N^{1,p}(X)} = 0$, is a Banach space, see [S00]. In this paper we assume that functions in $N^{1,p}(X)$ are defined everywhere (with values in $[-\infty, \infty]$), not just up to an equivalence class in the corresponding function space. Note that a modification of an $N^{1,p}(X)$ -function on a set of measure zero does not necessarily belong to $N^{1,p}(X)$.

The (Sobolev) *capacity* of an arbitrary set $E \subset X$ is

$$C_p(E) = \inf_u \|u\|_{N^{1,p}(X)}^p,$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \geq 1$ on E . A property holds *quasieverywhere* (q.e.) if the set of points for which the property does not hold has capacity zero. The capacity is the correct gauge for distinguishing between two Newtonian functions. If $u \in N^{1,p}(X)$, then $u \sim v$ if and only if $u = v$ q.e. Moreover, if $u, v \in N^{1,p}(X)$ and $u = v$ a.e., then $u = v$ q.e.

Definition 2.3. We say that X supports a p -Poincaré inequality if there exist constants $C > 0$ and $\lambda \geq 1$ such that for all balls $B \subset X$, all integrable functions f on X and all p -weak upper gradients g of f ,

$$\int_B |f - f_B| \, d\mu \leq C \operatorname{diam}(B) \left(\int_{\lambda B} g^p \, d\mu \right)^{1/p},$$

where $f_B := \int_B f \, d\mu / \mu(B)$.

From now on we assume that X supports a p -Poincaré inequality.

Let $\Omega \subset X$ be open. We define the *variational capacity* $\operatorname{cap}_p(E, \Omega)$ of $E \subset \Omega$ by

$$(2.1) \quad \operatorname{cap}_p(E, \Omega) = \inf_u \int_{\Omega} g_u^p \, d\mu,$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u = 1$ q.e. on E and $u = 0$ everywhere on $X \setminus \Omega$; we call such functions *admissible*. (One can equivalently assume that $u = 1$ (quasi)everywhere on E and $u = 0$ (quasi)everywhere on $X \setminus \Omega$.)

If there is an admissible function u (which happens if and only if $\text{cap}_p(E, \Omega) < \infty$), then there is also a minimizer of the problem (2.1) and it is unique up to sets of capacity zero. Moreover, there is a unique minimizer u_E^Ω which is also *lower semicontinuously regularized* in Ω , i.e.,

$$u_E^\Omega(x) := \text{ess} \liminf_{y \rightarrow x} u_E^\Omega(y) := \lim_{r \rightarrow 0} \left(\text{ess} \inf_{B(x,r)} u_E^\Omega \right), \quad x \in \Omega.$$

This unique minimizer u_E^Ω is the *capacitary potential* of E in Ω ; it is also referred to as the capacitary potential for $\text{cap}_p(E, \Omega)$. When it exists, the capacitary potential u_E^Ω satisfies

$$(2.2) \quad \text{cap}_p(E, \Omega) = \int_{\Omega} g_E^p d\mu,$$

where g_E is the minimal p -weak upper gradient of u_E^Ω . By definition $u_E^\Omega(x) = 0$ for $x \notin \Omega$.

Under our assumptions, (X, d) is L -quasiconvex, with L depending only on p and X . Here and below, when we say that a constant depends on p and X we really mean that it depends on p , the doubling constant and the constants in the p -Poincaré inequality. It follows from the quasiconvexity that the inner metric (as defined in (1.1)) is indeed a metric on X . Moreover, arc length for curves is the same with respect to d and d_{in} . Thus the class of p -weak upper gradients of a function is also the same with respect to both metrics, and as a consequence $N^{1,p}(X)$ is the same for both metrics. Moreover, (X, d_{in}, μ) satisfies our doubling and Poincaré assumptions, and thus the theory is directly applicable also with respect to d_{in} .

We say that two nonnegative quantities a and b are *comparable*, and write $a \simeq b$, if $a/C \leq b \leq Ca$ for some constant $C \geq 1$, where the constant C is referred to as the *constant of comparison*.

3. COMPARISON OF GLOBAL LOWER CAPACITY DENSITIES

We recall some well-known estimates for the capacity in balls.

Lemma 3.1 ([BB11, Proposition 6.16 and Lemma 11.22]). *Let $0 < a < b$. Then*

$$(3.1) \quad \text{cap}_p(B(x, ar), B(x, br)) \simeq r^{-p} \mu(B(x, r)),$$

where the constant of comparison depends only on a, b, p and X . Moreover, if $1 < s < t$, then

$$(3.2) \quad \text{cap}_p(E, B(x, tr)) \leq \text{cap}_p(E, B(x, sr)) \leq C \text{cap}_p(E, B(x, tr)) \quad \text{for } E \subset B(x, r),$$

where $C > 1$ depends only on s, t, p and X .

The corresponding estimates with respect to the inner metric also hold.

Using the estimates above, we can show that $\mathcal{D}(r, \tau, E)$ and $\mathcal{D}_{\text{in}}(r, \tau, E)$ are comparable in the following sense.

Lemma 3.2. *Let $\tau, \tau' > 1$. For every $r > 0$ and $E \subset X$ we have*

$$(3.3) \quad \mathcal{D}_{\text{in}}(r, \tau, E) \leq C \mathcal{D}(r, \tau, E) \leq C^2 \mathcal{D}_{\text{in}}(Lr, \tau, E),$$

where $C > 1$ depends only on τ, p and X , and L is the quasiconvexity constant. Moreover,

$$\mathcal{D}(r, \tau, E) \simeq \mathcal{D}(r, \tau', E) \quad \text{and} \quad \mathcal{D}_{\text{in}}(r, \tau, E) \simeq \mathcal{D}_{\text{in}}(r, \tau', E),$$

where the constants of comparison depend only on τ , τ' , p and X .

Proof. In view of (1.2) and Lemma 3.1 we see that

$$\begin{aligned} \text{cap}_p(E \cap B_{\text{in}}(x, r), B_{\text{in}}(x, \tau r)) &\simeq \text{cap}_p(E \cap B_{\text{in}}(x, r), B_{\text{in}}(x, \tau Lr)) \\ &\leq \text{cap}_p(E \cap B_{\text{in}}(x, r), B(x, \tau r)) \\ &\leq \text{cap}_p(E \cap B(x, r), B(x, \tau r)) \end{aligned}$$

and

$$\begin{aligned} \text{cap}_p(E \cap B(x, r), B(x, \tau r)) &\simeq \text{cap}_p(E \cap B(x, r), B(x, \tau Lr)) \\ &\leq \text{cap}_p(E \cap B_{\text{in}}(x, Lr), B(x, \tau Lr)) \\ &\leq \text{cap}_p(E \cap B_{\text{in}}(x, Lr), B_{\text{in}}(x, \tau Lr)), \end{aligned}$$

with constants of comparison depending only on τ , p and X . Hence (using (1.2) and (3.1) to see that the denominators are comparable),

$$\begin{aligned} \frac{\text{cap}_p(E \cap B_{\text{in}}(x, r), B_{\text{in}}(x, \tau r))}{\text{cap}_p(B_{\text{in}}(x, r), B_{\text{in}}(x, \tau r))} &\leq C' \frac{\text{cap}_p(E \cap B(x, r), B(x, \tau r))}{\text{cap}_p(B(x, r), B(x, \tau r))} \\ &\leq C'' \frac{\text{cap}_p(E \cap B_{\text{in}}(x, Lr), B_{\text{in}}(x, \tau Lr))}{\text{cap}_p(B_{\text{in}}(x, Lr), B_{\text{in}}(x, \tau Lr))}. \end{aligned}$$

Taking the infima with respect to $x \in X$ yields (3.3). The last assertion follows directly from (3.2) (and the corresponding estimate in the inner metric). \square

4. CAPACITY OF SUPERLEVEL SETS OF A CAPACITARY POTENTIAL

In this section we evaluate the capacity of superlevel sets of the capacitary potential, which may be of independent interest.

Proposition 4.1. *Let $E \subset \Omega$ with $\text{cap}_p(E, \Omega) < \infty$, and u_E be the capacitary potential of E in Ω . For $0 < M \leq 1$ let $E_M = \{x \in \Omega : u_E(x) > M\}$ and $E'_M = \{x \in \Omega : u_E(x) \geq M\}$. Then*

$$\begin{aligned} \text{cap}_p(E_M, \Omega) &= M^{1-p} \text{cap}_p(E, \Omega), \quad \text{if } 0 < M < 1, \\ \text{cap}_p(E'_M, \Omega) &= M^{1-p} \text{cap}_p(E, \Omega), \quad \text{if } 0 < M \leq 1. \end{aligned}$$

This result was obtained for weighted \mathbb{R}^n (with a p -admissible weight) in [HKM06, p. 118]. Their argument depends on the Euler–Lagrange equation, which is not available in the metric space setting considered here. Nevertheless, the weaker estimate

$$(4.1) \quad \text{cap}_p(E_M, \Omega) \simeq M^{1-p} \text{cap}_p(E, \Omega)$$

was obtained in [BMS01, Lemma 5.4] via a variational approach. Our proof of Proposition 4.1 is also based on the variational method, yet it yields the sharp identity in the metric space setting and is shorter than the earlier proofs of (4.1) and the proof in [HKM06, pp. 116–118].

Proof. For simplicity, write g_E for the minimal p -weak upper gradient of u_E . It follows from [BB11, Lemma 11.19] that

$$(4.2) \quad \text{cap}_p(E_M, \Omega) = \text{cap}_p(E'_M, \Omega) = \frac{1}{M^p} \int_{0 < u_E < M} g_E^p d\mu, \quad \text{if } 0 < M < 1.$$

The second equality in (4.2) also holds when $M = 1$ (and is easier to deduce than for $M < 1$). Hence, by (2.2), it suffices to show that

$$(4.3) \quad \int_{0 < u_E < M} g_E^p d\mu = M \int_{0 < u_E < 1} g_E^p d\mu.$$

For $0 < t < 1$ define the piecewise linear function $\Phi_t(s)$ on $[0, \infty)$ by

$$\Phi_t(s) = \begin{cases} \frac{ts}{M} & \text{for } 0 \leq s < M, \\ t + \frac{1-t}{1-M}(s-M) & \text{for } M \leq s < 1, \\ 1 & \text{for } s \geq 1. \end{cases}$$

We note that g_E vanishes a.e. on each level set $\{x \in \Omega : u_E(x) = t\}$. Therefore, for each $0 < t < 1$ we see that $v_t(x) := \Phi_t(u_E(x))$ is admissible for $\text{cap}_p(E, \Omega)$ and

$$\varphi(t) := \int_{\Omega} g_{v_t}^p d\mu = \left(\frac{t}{M}\right)^p \int_{0 < u_E < M} g_E^p d\mu + \left(\frac{1-t}{1-M}\right)^p \int_{M < u_E < 1} g_E^p d\mu.$$

By definition, $\Phi_M(s) = s$ for $0 \leq s \leq 1$, and so $v_M(x) = u_E(x)$. Hence

$$\varphi(t) \geq \varphi(M) = \int_{0 < u_E < 1} g_E^p d\mu = \text{cap}_p(E, \Omega)$$

with equality for $t = M$. In particular $\varphi'(M) = 0$. Since

$$\varphi'(t) = \frac{pt^{p-1}}{M^p} \int_{0 < u_E < M} g_E^p d\mu - \frac{p(1-t)^{p-1}}{(1-M)^p} \int_{M < u_E < 1} g_E^p d\mu,$$

it follows from $\varphi'(M) = 0$ that

$$\frac{1}{M} \int_{0 < u_E < M} g_E^p d\mu = \frac{1}{1-M} \int_{M < u_E < 1} g_E^p d\mu = \frac{1}{1-M} \left(\int_{0 < u_E < 1} g_E^p d\mu - \int_{0 < u_E < M} g_E^p d\mu \right),$$

which yields (4.3). \square

5. LOWER ESTIMATE OF CAPACITY DENSITY

We now use Proposition 4.1 to deduce estimates for the ratio of capacities in terms of the infimum of the corresponding capacity potential.

Lemma 5.1 (cf. [AI15, Lemma 4.2]). *Let $A, E \subset \Omega$ with $\text{cap}_p(A, \Omega) > 0$ and $\text{cap}_p(E, \Omega) < \infty$. Then the capacity potential u_E of E in Ω satisfies*

$$(5.1) \quad \inf_A u_E \leq \left(\frac{\text{cap}_p(E, \Omega)}{\text{cap}_p(A, \Omega)} \right)^{1/(p-1)}.$$

Proof. If $\inf_A u_E = 0$, then (5.1) holds trivially. Now suppose that $M = \inf_A u_E > 0$. Then $A \subset E'_M := \{x \in \Omega : u_E(x) \geq M\}$. Proposition 4.1 yields

$$\text{cap}_p(A, \Omega) \leq \text{cap}_p(E'_M, \Omega) = M^{1-p} \text{cap}_p(E, \Omega),$$

which readily gives the required inequality. \square

When A is a ball, there is a converse inequality to (5.1) up to a multiplicative constant depending only on p and X . Let $\Lambda = 100\lambda$ with $\lambda \geq 1$ being the dilation constant in the p -Poincaré inequality.

Lemma 5.2 ([BB11, Lemma 11.20]). *There exists $0 < C_0 \leq 1$, depending only on p and X , such that if $E \subset \overline{B(x, r)}$, then the capacitary potential u_E of E in $B(x, \Lambda r)$ satisfies*

$$\inf_{B(x, 2r)} u_E \geq C_0 \left(\frac{\text{cap}_p(E, B(x, \Lambda r))}{\text{cap}_p(B(x, 2r), B(x, \Lambda r))} \right)^{1/(p-1)}.$$

In metric spaces such an estimate was obtained in [BMS01, Lemma 5.6] and [B08, Lemma 3.9] under additional assumptions. For the reader's convenience, we sketch how this can be proved using Proposition 4.1.

Sketch of proof. Let $M = \sup_{\partial B(x, 2r)} u_E$ and let $E_M = \{x \in B(x, \Lambda r) : u_E(x) > M\}$. Then by the minimum principle (see [BB11, Proposition 9.4 and Theorem 9.13]), we get that $E_M \subset B(x, 2r)$. Hence by Proposition 4.1,

$$M^{1-p} \text{cap}_p(E, B(x, \Lambda r)) = \text{cap}_p(E_M, B(x, \Lambda r)) \leq \text{cap}_p(B(x, 2r), B(x, \Lambda r))$$

so that

$$M \geq \left(\frac{\text{cap}_p(E, B(x, \Lambda r))}{\text{cap}_p(B(x, 2r), B(x, \Lambda r))} \right)^{1/(p-1)}.$$

Finally using weak Harnack inequalities it can be shown that $\inf_{B(x, 2r)} u_E \geq C_0 M$, see the proof in [BB11]. \square

The following lemma is a variant of a comparison principle for capacitary potentials and will be useful when proving the subsequent results.

Lemma 5.3. *Let $V \subset \Omega$ be open and let $E' \subset E \subset \Omega$ be arbitrary sets such that $E' \subset V$, $\text{cap}_p(E', V) < \infty$ and $\text{cap}_p(E, \Omega) < \infty$. Let $u_{E'}^V$ and u_E^Ω be the corresponding capacitary potentials and assume that $1 - u_E^\Omega \leq a$ on ∂V with $0 \leq a \leq 1$. Then $1 - u_E^\Omega \leq a(1 - u_{E'}^V)$ in V .*

Proof. By the minimum principle, $u_E^\Omega \geq 1 - a$ in V . Hence

$$v := (u_E^\Omega - 1) + a \geq 0 \quad \text{in } V,$$

and it is easily verified that v is the lower semicontinuously regularized solution of the obstacle problem (see [BB11, Definition 7.1]) in V with the obstacle $a\chi_E$ and boundary data $v \geq 0$ on ∂V . Applying the comparison principle ([BB11, Lemma 8.30]) to v and $au_{E'}^V$ shows that

$$(u_E^\Omega - 1) + a = v \geq au_{E'}^V \quad \text{in } V,$$

from which the lemma follows. \square

For an open set U we let $\delta_U(x) = \text{dist}(x, X \setminus U)$ and define the ε -interior U_ε of U by

$$(5.2) \quad U_\varepsilon = \{x \in U : \delta_U(x) > \varepsilon\}.$$

Iterating Lemma 5.2, we obtain the following lemma.

Lemma 5.4. *Let $U \subset \Omega$ be open, $0 < \eta < 1$, and $r > 0$. Suppose that $E \subset U$ satisfies*

$$(5.3) \quad \frac{\text{cap}_p(E \cap B(x, r), B(x, \Lambda r))}{\text{cap}_p(B(x, 2r), B(x, \Lambda r))} \geq \eta \quad \text{for every } x \in U_{\Lambda r}.$$

If k is a positive integer and $U_{k\Lambda r} \neq \emptyset$, then

$$(5.4) \quad 1 - u_E^\Omega \leq (1 - C_0 \eta^{1/(p-1)})^k \quad \text{in } U_{k\Lambda r},$$

where $0 < C_0 \leq 1$ is as in Lemma 5.2.

Proof. Since $0 < C_0 \leq 1$ we see that $0 < 1 - C_0\eta^{1/(p-1)} < 1$. Take an arbitrary point $x \in U_{\Lambda r}$ and let $B = B(x, r)$. By Lemma 5.2 with $E \cap B$ in place of E and by (5.3) we see that the capacitary potential $u_{E \cap B}^{\Lambda B}$ of $E \cap B$ in ΛB satisfies

$$(5.5) \quad 1 - u_{E \cap B}^{\Lambda B} \leq 1 - C_0\eta^{1/(p-1)} \quad \text{in } B.$$

We prove (5.4) by induction on k using (5.5). Since $\delta_U(x) > \Lambda r$, we see that $\Lambda B \subset U \subset \Omega$, and hence by Lemma 5.3 and (5.5),

$$1 - u_E^\Omega \leq 1 - u_{E \cap B}^{\Lambda B} \leq 1 - C_0\eta^{1/(p-1)} \quad \text{in } B.$$

Since $x \in U_{\Lambda r}$ was arbitrary, we obtain (5.4) for $k = 1$.

Now let $k \geq 2$ and assume that (5.4) holds with $k - 1$ in place of k . Let $x \in U_{k\Lambda r}$ be arbitrary. Another application of Lemma 5.3 (with $V = \Lambda B$ and $a = (1 - C_0\eta^{1/(p-1)})^{k-1}$), together with (5.5), shows that

$$1 - u_E^\Omega \leq (1 - C_0\eta^{1/(p-1)})^{k-1} (1 - u_{E \cap B}^{\Lambda B}) \leq (1 - C_0\eta^{1/(p-1)})^k \quad \text{in } B,$$

which, due to the arbitrariness of $x \in U_{k\Lambda r}$, amounts to (5.4). This completes the induction. \square

Lemmas 5.4 and 5.1 (the latter with $U_{k\Lambda r}$ and $E \cap U$ in place of A and E) readily give the following lower bound for the ratio of capacities. (For $x \in U_{\Lambda r}$ we have $E \cap U \cap B(x, r) = E \cap B(x, r)$ so that (5.3) holds with $E \cap U$ in place of E for such x .)

Corollary 5.5. *Let $U \subset \Omega$ be open, $0 < \eta < 1$ and $r > 0$. Suppose that $E \subset X$ satisfies (5.3). If k is a positive integer and $U_{k\Lambda r} \neq \emptyset$, then*

$$\frac{\text{cap}_p(E \cap U, \Omega)}{\text{cap}_p(U_{k\Lambda r}, \Omega)} \geq (1 - (1 - C_0\eta^{1/(p-1)})^k)^{p-1}.$$

Remark 5.6. Results analogous to those in this section for the inner metric follow immediately, as seen from the discussion in the penultimate paragraph of Section 2.

In the next section, we shall see that, if R is large, then $\text{cap}_p(B_{\text{in}}(x, R - k\Lambda r), B_{\text{in}}(x, \tau R))$ is close to $\text{cap}_p(B_{\text{in}}(x, R), B_{\text{in}}(x, \tau R))$ uniformly for $x \in X$. This property does not hold for ordinary balls. This is the reason why $\mathcal{D}_{\text{in}}(r, \tau, E)$ has dichotomy and yet $\mathcal{D}(r, \tau, E)$ does not.

6. UNIFORM APPROXIMATION OF CAPACITY FROM INSIDE AND PROOF OF THEOREM 1.1

Let U be an open set and recall from (5.2) that $U_\varepsilon = \{x \in U : \delta_U(x) > \varepsilon\}$ is the ε -interior of U . We also define the ε -neighborhood of U by

$$(6.1) \quad U[\varepsilon] = \{x \in \mathbb{R}^n : \text{dist}(x, U) < \varepsilon\}.$$

The main aim of this section is to prove Theorem 1.1. In order to do so we will show that the capacity of U_ε approximates the capacity of U , under suitable assumptions on U .

Definition 6.1. Let $0 < \kappa < 1$ and $0 \leq R_1 < R_2$. We say that U satisfies the *interior corkscrew condition* with parameters κ , R_1 and R_2 if

$$x \in U \text{ and } R_1 < r < R_2 \implies U \cap B(x, r) \text{ contains a ball of radius } \kappa r.$$

Remark 6.2. For $R > 0$, $B_{\text{in}}(x, R)$ satisfies the interior corkscrew condition with parameters $1/2L$, 0 and R . The same is not true in general for ordinary balls, cf. Proposition 7.1.

Lemma 6.3. *Suppose that U satisfies the interior corkscrew condition with parameters κ , 0 and R_2 . Let $0 < \varepsilon < \kappa R_2/2$. Then:*

- (i) *For every $x \in U$ and $2\varepsilon/\kappa \leq r < R_2$, the set $U_\varepsilon \cap B(x, r)$ contains a ball of radius $\kappa r/2$. In particular, U_ε satisfies the interior corkscrew condition with parameters $\kappa/2$, $2\varepsilon/\kappa$ and R_2 .*
- (ii) *$U \subset U_\varepsilon[2\varepsilon/\kappa]$.*

Proof. (i) Let $x \in U$ and $2\varepsilon/\kappa \leq r < R_2$. By hypothesis there is a ball $B(y, \kappa r) \subset U \cap B(x, r)$. This means that $\delta_U(y) \geq \kappa r \geq 2\varepsilon$, so that

$$\delta_{U_\varepsilon}(y) \geq \delta_U(y) - \varepsilon \geq \kappa r - \frac{\kappa r}{2} = \frac{\kappa r}{2}.$$

Hence $B(y, \kappa r/2) \subset U_\varepsilon \cap B(x, r)$.

(ii) Let $x \in U$ and apply (i) with $r = 2\varepsilon/\kappa$. We find a ball $B(y, \varepsilon) \subset U_\varepsilon \cap B(x, 2\varepsilon/\kappa)$. Then $y \in U_\varepsilon$ and $d(x, y) < 2\varepsilon/\kappa$, so that $x \in U_\varepsilon[2\varepsilon/\kappa]$. \square

Lemma 6.4. *Suppose that $U \subset \Omega$ satisfies the interior corkscrew condition with parameters κ , R_1 and $R_2 \leq \text{dist}(U, X \setminus \Omega)$. If $j \geq 1$ and $R_2/\Lambda^j > R_1$, then*

$$\frac{\text{cap}_p(U[R_2/\Lambda^j], \Omega)}{\text{cap}_p(U, \Omega)} \leq (1 - \eta_\kappa^j)^{1-p},$$

where $0 < \eta_\kappa < 1$ depends only on κ , p and X .

Proof. Let $x \in U$ be arbitrary. In view of Lemma 3.1, we find $0 < \eta < 1$ depending only on κ , p and X such that if $x \in U$, then

$$\frac{\text{cap}_p(U \cap B(x, r), B(x, \Lambda r))}{\text{cap}_p(B(x, 2r), B(x, \Lambda r))} \geq \eta \quad \text{for all } R_1 < r < R_2.$$

This, together with Lemma 5.2, yields for $R_1 < r < R_2$,

$$(6.2) \quad 1 - u_r \leq 1 - C_0 \eta^{1/(p-1)} =: \eta_\kappa \quad \text{in } B(x, r),$$

where u_r is the capacitary potential of $U \cap B(x, r)$ in $B(x, \Lambda r)$ and $0 < \eta_\kappa < 1$ depends only on κ , p and X . Let u_U be the capacitary potential of U in Ω . We shall show that (6.2) implies

$$(6.3) \quad 1 - u_U \leq \eta_\kappa^j \quad \text{in } B(x, R_2/\Lambda^j)$$

whenever $R_2/\Lambda^j > R_1$. To start with, note that Lemma 5.3 (with $V = B(x, R_2/\Lambda)$ and $a = 1$) and (6.2) imply

$$1 - u_U \leq 1 - u_{R_2/\Lambda} \leq \eta_\kappa \quad \text{in } B(x, R_2/\Lambda),$$

i.e., (6.3) holds for $j = 1$. Now let $j \geq 2$ and assume that (6.3) holds with j replaced by $j-1$. As $R_2/\Lambda^j > R_1$, we know that (6.2) holds for $r = R_2/\Lambda^j$. Now, applying Lemma 5.3 with $V = B(x, R_2/\Lambda^{j-1})$ and $a = \eta_\kappa^{j-1}$, yields

$$1 - u_U \leq \eta_\kappa^{j-1} (1 - u_{R_2/\Lambda^j}) \leq \eta_\kappa^j \quad \text{in } B(x, R_2/\Lambda^j),$$

which proves (6.3) also for j . Since $x \in U$ was arbitrary, we conclude that

$$U[R_2/\Lambda^j] \subset \{x \in \Omega : u_U(x) \geq 1 - \eta_\kappa^j\}.$$

Hence Proposition 4.1 yields the required inequality. \square

By Lemmas 6.3 and 6.4 with U_ε in place of U we immediately obtain the following approximation of capacity from inside.

Lemma 6.5. *Suppose that $U \subset \Omega$ satisfies the interior corkscrew condition with parameters κ , 0 and $R_2 \leq \text{dist}(U, X \setminus \Omega)$. Let $0 < \eta_{\kappa/2} < 1$ be the constant in Lemma 6.4 corresponding to $\kappa/2$. If $j \geq 1$ and $\varepsilon \leq \kappa R_2 / 2\Lambda^j$, then*

$$\frac{\text{cap}_p(U, \Omega)}{\text{cap}_p(U_\varepsilon, \Omega)} \leq (1 - \eta_{\kappa/2}^j)^{1-p}.$$

Proof of Theorem 1.1. In view of Lemma 3.2, it is sufficient to show that if $\mathcal{D}(r, \Lambda, E) > 0$ for some $r > 0$, then $\lim_{R \rightarrow \infty} \mathcal{D}_{\text{in}}(R, \tau, E) = 1$. Note that $\mathcal{D}(r, \Lambda, E) > 0$ implies (5.3) for some $0 < \eta < 1$. Take an arbitrary positive number $\alpha < 1$ and find a positive integer k such that

$$(1 - (1 - C_0 \eta^{1/(p-1)})^k)^{p-1} \geq \alpha,$$

where C_0 is the constant from Corollary 5.5. By Remark 6.2, $B_{\text{in}} := B_{\text{in}}(x, R)$ satisfies the corkscrew condition with parameters $\kappa = 1/2L$, 0 and $R_2 = \min\{1, \tau - 1\}R$. Corollary 5.5, together with Lemma 6.5 (and $U = B_{\text{in}}$, $\Omega = \tau B_{\text{in}} = B_{\text{in}}(x, \tau R)$ and $\varepsilon = k\Lambda r$), then implies that

$$\mathcal{D}_{\text{in}}(R, \tau, E) = \inf_{x \in X} \frac{\text{cap}_p(E \cap B_{\text{in}}, \tau B_{\text{in}})}{\text{cap}_p((B_{\text{in}})_{k\Lambda r}, \tau B_{\text{in}})} \frac{\text{cap}_p((B_{\text{in}})_{k\Lambda r}, \tau B_{\text{in}})}{\text{cap}_p(B_{\text{in}}, \tau B_{\text{in}})} \geq \alpha (1 - \eta_{\kappa/2}^j)^{p-1},$$

where j is the maximal integer such that

$$k\Lambda r \leq \frac{\kappa \min\{1, \tau - 1\}R}{2\Lambda^j}.$$

Letting $R \rightarrow \infty$ (and thus $j \rightarrow \infty$) and then $\alpha \rightarrow 1$ shows that $\lim_{R \rightarrow \infty} \mathcal{D}_{\text{in}}(R, \tau, E) = 1$, since clearly $\mathcal{D}_{\text{in}}(R, \tau, E) \leq 1$ for all $R > 0$. \square

Proof of Theorem 1.3. This follows directly from Theorem 1.1 and Lemma 3.2. \square

7. COUNTEREXAMPLES AND PROOF OF THEOREM 1.2

In this section we shall first construct an example (X, d, μ) for which the dichotomy for ordinary balls does not hold. Let $B^+(x, r) = \{y \in B(x, r) : y^n > x^n\}$ with $x = (x^1, \dots, x^n) \in \mathbb{R}^n$. This is the open upper half ball in \mathbb{R}^n with center at x and radius r . The half-open lower half ball is denoted by $B^-(x, r) := B(x, r) \setminus B^+(x, r)$.

Let $x_j = (4^j, 0, \dots, 0)$ and $R_j = 2^j$, $j = 1, 2, \dots$. Let $X = \mathbb{R}^n \setminus \bigcup_{j=1}^\infty B^+(x_j, R_j)$ and let $d(x, y)$ be the restriction of the Euclidean distance to X . We write $B_X(x, r) = \{y \in X : d(x, y) < r\}$ for the open ball with center at x and radius r in X with respect to $d(x, y)$. Observe that $B_X(x, r) = B(x, r) \cap X$ with $B(x, r)$ being the Euclidean ball with center at x and radius r . Let μ be the restriction of n -dimensional Lebesgue measure on X . Then μ is doubling on X . Moreover, X is the closure of a uniform domain in \mathbb{R}^n and hence supports a 1-Poincaré inequality, by [BS07, Theorem 4.4] and [AS05, Proposition 7.1]. We will denote the variational capacities with respect to X and \mathbb{R}^n by cap_p and $\text{cap}_p^{\mathbb{R}^n}$, respectively.

Proposition 7.1. *Let $1 < p < n$ and $\tau > 1$. In the situation described above the following assertions hold true:*

- (i) The balls $B_X(x, r)$ fail the uniform approximation of capacity. More precisely, if $\rho > 0$, then $R_j/(R_j + \rho) \uparrow 1$, as $j \rightarrow \infty$, and yet for $2^j \geq 4 \max\{\tau, \rho\}$,

$$\frac{\text{cap}_p(B_X(x_j, R_j), B_X(x_j, \tau(R_j + \rho)))}{\text{cap}_p(B_X(x_j, R_j + \rho), B_X(x_j, \tau(R_j + \rho)))} \leq C < 1,$$

where C is independent of ρ .

- (ii) No dichotomy property, with respect to the balls $B_X(x, r)$, holds for the set

$$(7.1) \quad E = \bigcup_{z \in \mathbb{Z}^n \setminus H} B(z, \delta), \quad \text{where } 0 < \delta \leq \frac{1}{4}, \quad H = \bigcup_{j=1}^{\infty} B^+(x_j, R_j) \left[\frac{1}{2}\right]$$

and $B^+(x_j, R_j) \left[\frac{1}{2}\right]$ is the $\frac{1}{2}$ -neighborhood of $B^+(x_j, R_j)$, here taken with respect to \mathbb{R}^n , see (6.1) and Figure 1. More precisely,

- (a) $\mathcal{D}(2\sqrt{n}, \tau, E) > 0$,
(b) $0 < \liminf_{R \rightarrow \infty} \mathcal{D}(R, \tau, E) < 1$.

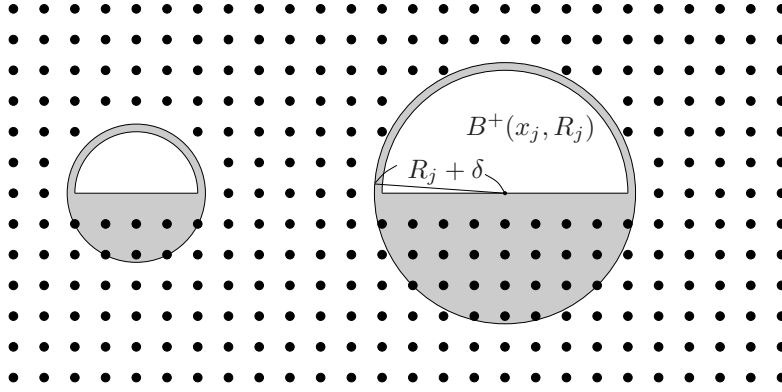


FIGURE 1. No dichotomy holds for $X = \mathbb{R}^n \setminus \bigcup_{j=1}^{\infty} B^+(x_j, R_j)$ with E being the union of all small black balls.

Proof. From the construction, the balls $\{B(x_j, \frac{3}{5}4^j)\}_{j=1}^{\infty}$ are pairwise disjoint. To prove (i) let $\rho > 0$ and $2^j \geq 4 \max\{\tau, \rho\}$. Then $B(x_j, \tau(R_j + \rho)) \subset B(x_j, \frac{3}{5}4^j)$, and thus $B(x_j, \tau(R_j + \rho))$ does not intersect any of the balls $B(x_k, R_k)$, $k \neq j$. Hence

$$(7.2) \quad \begin{aligned} \text{cap}_p(B_X(x_j, R_j), B_X(x_j, \tau R_j)) &\leq \text{cap}_p^{\mathbb{R}^n}(B^-(x_j, R_j), B(x_j, \tau R_j)), \\ \text{cap}_p(B_X(x_j, R_j + \rho), B_X(x_j, \tau(R_j + \rho))) &= \text{cap}_p^{\mathbb{R}^n}(B(x_j, R_j + \rho), B(x_j, \tau(R_j + \rho))), \end{aligned}$$

which, together with translation and dilation for $\text{cap}_p^{\mathbb{R}^n}$, yields

$$(7.3) \quad \begin{aligned} \frac{\text{cap}_p(B_X(x_j, R_j), B_X(x_j, \tau R_j))}{\text{cap}_p(B_X(x_j, R_j + \rho), B_X(x_j, \tau(R_j + \rho)))} &\leq \frac{R_j^{n-p} \text{cap}_p^{\mathbb{R}^n}(B^-(0, 1), B(0, \tau))}{(R_j + \rho)^{n-p} \text{cap}_p^{\mathbb{R}^n}(B(0, 1), B(0, \tau))} \\ &\leq \frac{\text{cap}_p^{\mathbb{R}^n}(B^-(0, 1), B(0, \tau))}{\text{cap}_p^{\mathbb{R}^n}(B(0, 1), B(0, \tau))} =: C < 1. \end{aligned}$$

Thus (i) follows.

For the proof of (ii), let $0 < \delta \leq \frac{1}{4}$ and note that if $x \in X$, then there exists $x' \in X \setminus H$ such that $d(x, x') \leq \frac{1}{2}$. Now, by going at most length 1 in each of the coordinate directions,

we can find $z \in \mathbb{Z}^n \cap (X \setminus H)$ such that $d(x', z) \leq \sqrt{n}$. It thus follows from Lemma 3.1 that

$$\begin{aligned} \text{cap}_p(E \cap B_X(x, 2\sqrt{n}), B_X(x, 2\tau\sqrt{n})) &\geq \text{cap}_p(B_X(z, \delta), B_X(z, 4\tau\sqrt{n})) \\ &\geq C' \text{cap}_p(B_X(z, \delta), B_X(z, \tfrac{1}{2})) = C' \text{cap}_p^{\mathbb{R}^n}(B(z, \delta), B(z, \tfrac{1}{2})) \geq C'' \delta^{n-p}, \end{aligned}$$

where C' and C'' depend only on n, p and τ . Taking infimum over $x \in X$, we obtain (a). It then follows from Lemma 3.2 and Theorem 1.1 that

$$\liminf_{R \rightarrow \infty} \mathcal{D}(R, \tau, E) \geq C''' \liminf_{R \rightarrow \infty} \mathcal{D}_{\text{in}}(R, \tau, E) = C''' > 0,$$

where C''' depends only on n, p and τ . By (7.1) we have

$$E \cap B_X(x_j, R_j + \delta) \subset B^-(x_j, R_j + \delta).$$

Moreover, if $2^j \geq 4\tau$, then (7.2) with $\rho = \delta$ yields as in (7.3),

$$\frac{\text{cap}_p(E \cap B_X(x_j, R_j + \delta), B_X(x_j, \tau(R_j + \delta)))}{\text{cap}_p(B_X(x_j, R_j + \delta), B_X(x_j, \tau(R_j + \delta)))} \leq \frac{\text{cap}_p^{\mathbb{R}^n}(B^-(0, 1), B(0, \tau))}{\text{cap}_p^{\mathbb{R}^n}(B(0, 1), B(0, \tau))} < 1.$$

Hence

$$\mathcal{D}(R_j + \delta, \tau, E) \leq \frac{\text{cap}_p^{\mathbb{R}^n}(B^-(0, 1), B(0, \tau))}{\text{cap}_p^{\mathbb{R}^n}(B(0, 1), B(0, \tau))} < 1,$$

so that $\liminf_{R \rightarrow \infty} \mathcal{D}(R, \tau, E) < 1$. Thus (b) is proved. \square

The following example shows that the Sobolev capacity C_p has no dichotomy nor a weak dichotomy similar to the one in Theorem 1.3. Define

$$\mathcal{D}^{C_p}(r, E) = \inf_{x \in X} \frac{C_p(E \cap B(x, r))}{C_p(B(x, r))}.$$

We are interested in the behavior of $\mathcal{D}^{C_p}(r, E)$ as $r \rightarrow \infty$.

Example 7.2. Let $X = \mathbb{R}^n$ (unweighted) and $1 < p < \infty$. Note that $\mu(E) \leq C_p(E)$ for every measurable set E . For $B(x, r)$ and $r \geq 1$ we can test the capacity with $u(y) = (1 - \text{dist}(y, B(x, r)))_+$, which shows that

$$(7.4) \quad r^n \omega_n = \mu(B(x, r)) \leq C_p(B(x, r)) \leq 2 \cdot (2r)^n \omega_n = 2^{n+1} r^n \omega_n,$$

where $\omega_n = \mu(B(0, 1))$. Let $M \geq 10$, $A = (M\mathbb{Z})^n = \{\dots, -M, 0, M, \dots\}^n$ and $E_M = \bigcup_{z \in A} B(z, 1)$. Also let $x \in X$.

Using (7.4) with $r = 1$ and estimating the number of balls $B(z, 1)$ in $E_M \cap B(x, r)$, $r \geq 10M$, gives

$$\left(\frac{r}{M\sqrt{n}}\right)^n \omega_n \leq \mu(B(x, r) \cap E_M) \leq C_p(B(x, r) \cap E_M) \leq \left(\frac{3r}{M}\right)^n 2^{n+1} \omega_n = 2 \left(\frac{6r}{M}\right)^n \omega_n.$$

Combining this estimate with (7.4) shows that

$$\frac{1}{2(2M\sqrt{n})^n} \leq \liminf_{r \rightarrow \infty} \mathcal{D}^{C_p}(r, E_M) \leq \limsup_{r \rightarrow \infty} \mathcal{D}^{C_p}(r, E_M) \leq 2 \left(\frac{6}{M}\right)^n.$$

It follows that, by varying M , $\liminf_{r \rightarrow \infty} \mathcal{D}^{C_p}(r, E_M)$ can take at least a countable number of different values in the interval $[0, 1]$, including the end points since $\mathcal{D}^{C_p}(r, X) = 1$ and $\mathcal{D}^{C_p}(r, \emptyset) = 0$ for all r . Most likely it can take any value in the interval.

8. DICHOTOMY AND CAPACITARILY STABLE COLLECTIONS

In studying the proof of Theorem 1.1, it turns out that dichotomy holds for many more families of sets than the family of inner balls. In this section we first extract the key properties such a family might have and then demonstrate dichotomy under these assumptions. We then proceed to give examples of such capacitarily stable families.

Definition 8.1. A collection \mathfrak{U} of bounded open subsets of X is *capacitarily stable* if there exist constants $\tau > 1$, $\gamma \geq 1$ and a function $\varphi : (0, \infty)^2 \rightarrow (0, 1]$ such that:

- (i) For every ball $B \subset X$ we can find $U \in \mathfrak{U}$ such that $B \subset U \subset \gamma B$.
- (ii) For each $U \in \mathfrak{U}$ there exists a ball $B_U \subset X$ such that $B_U \subset U \subset \gamma B_U$.
- (iii) For every $\rho, R > 0$ and every $U \in \mathfrak{U}$ with $\text{diam}(U) \geq R$ we have

$$\frac{\text{cap}_p(U_\rho, U^*)}{\text{cap}_p(U, U^*)} \geq 1 - \varphi(\rho, R),$$

where U_ρ is the ρ -interior of U as in (5.2), and $U^* := \tau\gamma B_U$.

- (iv) For every $\rho > 0$,

$$\lim_{R \rightarrow \infty} \varphi(\rho, R) = 0.$$

Definition 8.2. Given a capacitarily stable collection \mathfrak{U} with parameters τ , γ and φ , we set for $r > 0$ and $E \subset X$,

$$\mathcal{D}^{\mathfrak{U}}(r, E) = \inf_{\substack{U \in \mathfrak{U} \\ r \leq \text{diam}(U) \leq 2\gamma r}} \frac{\text{cap}_p(E \cap U, U^*)}{\text{cap}_p(U, U^*)}.$$

Note that since X (under our assumptions) is connected and unbounded we have that $r \leq \text{diam}(B(x, r)) \leq 2r$ for every ball $B(x, r)$. Hence, because of (i), the collection $\{U \in \mathfrak{U} : r \leq \text{diam}(U) \leq 2\gamma r\}$ is nonempty, and thus $\mathcal{D}^{\mathfrak{U}}(r, E) < \infty$ (and so ≤ 1). A capacitarily stable collection \mathfrak{U} might be associated with more than one choice of the parameters τ and γ . Different choices of τ and γ impact the value of $\mathcal{D}^{\mathfrak{U}}(r, E)$. However, the value of $\mathcal{D}^{\mathfrak{U}}(r, E)$ is independent of the choice of φ .

We are now ready to obtain the main dichotomy result for capacitarily stable collections. Since X is unbounded it follows from Definition 8.1 (i) that $\sup_{U \in \mathfrak{U}} \text{diam}(U) = \infty$ whenever \mathfrak{U} is a capacitarily stable collection, and thus it makes sense to consider the limits $R \rightarrow \infty$ in Theorem 8.3 and Corollary 8.4 below.

Theorem 8.3. Let \mathfrak{U} and \mathfrak{U}' be capacitarily stable collections of bounded open sets in X , $\tau > 1$, and $E \subset X$. Then the following statements are equivalent:

- (a) $\mathcal{D}^{\mathfrak{U}}(r, E) > 0$ for some $r > 0$,
- (b) $\lim_{R \rightarrow \infty} \mathcal{D}^{\mathfrak{U}}(R, E) = 1$,
- (c) $\lim_{R \rightarrow \infty} \mathcal{D}^{\mathfrak{U}'}(R, E) = 1$,
- (d) $\lim_{R \rightarrow \infty} \mathcal{D}_{\text{in}}(R, \tau, E) = 1$.

As an immediate corollary we obtain the following dichotomy.

Corollary 8.4. Let \mathfrak{U} be a capacitarily stable collection of bounded open sets in X . Then for every $E \subset X$ one of the following statements holds:

- (i) $\lim_{R \rightarrow \infty} \mathcal{D}^{\mathfrak{U}}(R, E) = 0$,
- (ii) $\lim_{R \rightarrow \infty} \mathcal{D}^{\mathfrak{U}}(R, E) = 1$.

Furthermore, these two possibilities are independent of \mathfrak{U} and its associated parameters.

Note also that by appealing to Theorem 1.1 we can directly obtain several further statements equivalent to those in Theorem 8.3.

For the dichotomy to hold what happens at small scales is irrelevant. We could therefore have associated yet another parameter $R_0 \geq 0$ with capacitarly stable collections, requiring (i) and (iii) in Definition 8.1 to hold only for $\text{diam}(B) > R_0$ resp. $R > R_0$. The implications (a) \Rightarrow (b), (c), (d) in Theorem 8.3 would then hold provided that r is sufficiently large (depending on R_0). A drawback would however have been that here, as well as in results similar to Theorems 8.5 and 8.6, one also would have to consider possible enlargements of this parameter. We have refrained from this generalization.

Proof of Theorem 8.3. To facilitate the proof we introduce one more statement that will be shown to be equivalent to the statements in the theorem:

(e) $\mathcal{D}(r, \Lambda, E) > 0$ for some $r > 0$.

Recall that $\Lambda = 100\lambda$, where $\lambda \geq 1$ is the dilation constant in the p -Poincaré inequality.

(b) \Rightarrow (a) This is trivial.

(a) \Rightarrow (e) It is sufficient to prove that for all $r > 0$,

$$(8.1) \quad C\mathcal{D}^{\mathfrak{U}}(r, E) \leq \mathcal{D}(\gamma r, \Lambda, E),$$

where $C > 0$ depends only on the parameters of \mathfrak{U} . Let $r > 0$ and $x \in X$. By Definition 8.1 (i) we find $U \in \mathfrak{U}$ such that $B(x, r) \subset U \subset B(x, \gamma r)$, and then by Definition 8.1 (ii) we find a ball $B_U = B(x_U, r_U)$ such that $B_U \subset U \subset \gamma B_U$. As $x_U \in U \subset B(x, \gamma r)$, we see that $d(x, x_U) < \gamma r$, so that $B(x, \Lambda \gamma r) \subset B(x_U, (\Lambda + 1)\gamma r)$. Similarly, $d(x, x_U) < \gamma r_U$ and thus $U^* = B(x_U, \tau \gamma r_U) \subset B(x, (\tau + 1)\gamma r_U)$. Hence Lemma 3.1 shows that

$$\text{cap}_p(E \cap B(x, \gamma r), B(x, \Lambda \gamma r)) \geq \text{cap}_p(E \cap U, B(x_U, (\Lambda + 1)\gamma r)) \simeq \text{cap}_p(E \cap U, U^*)$$

and

$$\text{cap}_p(U, U^*) \geq \text{cap}_p(B(x, r), B(x, (\tau + 1)\gamma r_U)) \simeq \text{cap}_p(B(x, \gamma r), B(x, \Lambda \gamma r)).$$

Since $r \leq \text{diam}(U) \leq 2\gamma r$, we get that

$$\frac{\text{cap}_p(E \cap B(x, \gamma r), B(x, \Lambda \gamma r))}{\text{cap}_p(B(x, \gamma r), B(x, \Lambda \gamma r))} \geq C \frac{\text{cap}_p(E \cap U, U^*)}{\text{cap}_p(U, U^*)} \geq C\mathcal{D}^{\mathfrak{U}}(r, E).$$

Taking the infimum with respect to $x \in X$, we obtain (8.1).

(e) \Rightarrow (b) The proof of this implication is almost the same as the proof of Theorem 1.1. Let $r > 0$ be such that $\mathcal{D}(r, \Lambda, E) > 0$. Note that this implies the hypothesis (5.3) of Lemma 5.4 for some $0 < \eta < 1$. Let $U \in \mathfrak{U}$ and let U^* be as in Definition 8.1 (iii). Corollary 5.5 then gives

$$\frac{\text{cap}_p(E \cap U, U^*)}{\text{cap}_p(U_{k\Lambda r}, U^*)} \geq (1 - (1 - C_0 \eta^{1/(p-1)})^k)^{p-1},$$

whenever $U_{k\Lambda r} \neq \emptyset$.

Take an arbitrary positive number $\alpha < 1$ and find a positive integer k such that the right-hand side of the above inequality is greater than α . If $\text{diam}(U) \geq R$ and $B_U = B(x_U, r_U)$ is as in Definition 8.1 (ii), then $R \leq 2\gamma r_U$ and $x_U \in U_{k\Lambda r}$ provided that $k\Lambda r < r_U$. Thus, $U_{k\Lambda r} \neq \emptyset$ whenever $\text{diam}(U) \geq R > 2\gamma k\Lambda r$. Definition 8.1 (iii) with $\rho = \gamma k\Lambda r$

then yields that for $R > 2\rho$,

$$\mathcal{D}^{\mathfrak{U}}(R, E) \geq \inf_{\substack{U \in \mathfrak{U} \\ \text{diam}(U) \geq R}} \frac{\text{cap}_p(E \cap U, U^*)}{\text{cap}_p(U_\rho, U^*)} \frac{\text{cap}_p(U_\rho, U^*)}{\text{cap}_p(U, U^*)} \geq \alpha(1 - \varphi(\rho, R)).$$

Letting $R \rightarrow \infty$ and then $\alpha \rightarrow 1$ shows that $\lim_{R \rightarrow \infty} \mathcal{D}^{\mathfrak{U}}(R, E) = 1$, by Definition 8.1 (iv).

(c) \Leftrightarrow (e) As we have now shown that (b) \Leftrightarrow (e), swapping the roles of \mathfrak{U} and \mathfrak{U}' immediately yields (c) \Leftrightarrow (e).

(d) \Leftrightarrow (e) This follows directly from Theorem 1.1. \square

Next, we will present several useful examples of capacitarly stable collections.

Theorem 8.5. *Assume that \mathfrak{U} is a family of open subsets of X which satisfies Definition 8.1 (i) with $\gamma \geq 1$, and that there exists $\beta > 0$ such that every $U \in \mathfrak{U}$ satisfies the interior corkscrew condition with parameters κ , 0 and $\beta \text{diam}(U)$. Let $\tau > 1$ and $\tilde{\gamma} := \max\{\gamma, 1/\kappa\beta\}$. Then there is a function φ such that \mathfrak{U} is capacitarly stable with parameters τ , $\tilde{\gamma}$ and φ .*

In view of Remark 6.2, it follows in particular that the family \mathfrak{U} of all inner balls is capacitarly stable, however, note that the density $\mathcal{D}^{\mathfrak{U}}$ is obtained by looking at inner balls of diameters between r and $2\gamma r$, while \mathcal{D}_{in} is obtained by looking at inner balls of radius r ; thus these two numbers could be different for each $r > 0$.

Proof. For $U \in \mathfrak{U}$, pick $x \in U$ and use the corkscrew condition to find a ball

$$B_U := B(z_U, \kappa\beta \text{diam}(U)) \subset U \cap B(x, \beta \text{diam}(U)).$$

Then (ii) holds with $\tilde{\gamma}$.

To prove (iii) and (iv), let $\rho, R > 0$, set $R_2 = \min\{\beta, \tau - 1\}R$ and let j be the largest integer such that

$$\Lambda^j \leq \frac{\kappa R_2}{2\rho}.$$

Given $U \in \mathfrak{U}$ with $\text{diam}(U) \geq R$, Lemma 6.5 with $\Omega = U^* := \tau\tilde{\gamma}B_U$ implies that

$$\frac{\text{cap}_p(U_\rho, U^*)}{\text{cap}_p(U, U^*)} \geq (1 - \eta_{\kappa/2}^{j_0})^{p-1} \geq 1 - \max\{1, p-1\}\eta_{\kappa/2}^{j_0} =: 1 - \varphi(\rho, R),$$

where $j_0 = \max\{j, 0\}$. Since $j_0 \geq \log R / \log \Lambda + a$ for some constant a depending on κ , β , τ , Λ and ρ , this implies that for every fixed ρ , we have $\varphi(\rho, R) \rightarrow 0$ as $R \rightarrow \infty$, i.e. (iv) holds. \square

Theorem 8.6. *Let \mathfrak{U} be a family of open sets satisfying (i) and (ii) of Definition 8.1. Let $\beta > 0$. For each $U \in \mathfrak{U}$, set*

$$U^\beta = \{x \in X : \text{dist}_{\text{in}}(x, U) < \beta \text{diam}(U)\}.$$

Then $\mathfrak{U}^\beta := \{U^\beta : U \in \mathfrak{U}\}$ is capacitarly stable.

Proof. It can be shown as in the proof of Proposition 3.4 in [BB07] that each U^β satisfies the interior corkscrew condition with parameters $\kappa = 1/3L$, 0 and $\beta \text{diam}(U)/3$, where L is the quasiconvexity constant. Also, by (i) for \mathfrak{U} , if $B = B(x, r)$, then there is $U \in \mathfrak{U}$ such that $B \subset U \subset \gamma B$. Since

$$\text{diam}(U^\beta) \leq (1 + 2\beta) \text{diam}(U) \leq (1 + 2\beta)2\gamma r =: \tilde{\gamma}r,$$

we see that $B \subset U^\beta \subset \tilde{\gamma}B$. Thus, (i) holds for \mathfrak{U}^β as well and Theorem 8.5 concludes the proof. \square

Remark 8.7. A particularly well shaped collection \mathfrak{U}^β is obtained if

$$\mathfrak{U} = \{B(x, r) : x \in X \text{ and } r > 0\}.$$

Then the “almost balls” $B^\beta(x, r)$ satisfy $B(x, r) \subset B^\beta(x, r) \subset B(x, (1 + 2\beta)r)$ and are thus closer in shape to ordinary balls than what inner balls are, cf. (1.2). By Theorems 8.3 and 8.6, dichotomy holds for these “almost balls”. If X is geodesic, i.e. for inner balls, we have $B^\beta(x, r) = B(x, (1 + 2\beta)r)$.

The inner balls are John domains, see Remark 8.11 below. It is therefore natural to study dichotomy for John domains.

Definition 8.8. For an open set U we let $\delta_U(x) = \text{dist}(x, X \setminus U)$. Let $0 < c_J \leq 1$. We say that U is a c_J -John domain if there exists (a John center) $x_U \in U$ such that each $x \in U$ can be connected to x_U by a rectifiable curve $\gamma \subset U$ with

$$(8.2) \quad c_J \ell(\gamma(x, y)) \leq \delta_U(y) \quad \text{for all } y \in \gamma,$$

where $\gamma(x, y)$ is the subcurve of γ from x to y . Such a curve γ will be referred to as a c_J -John curve connecting x and x_U .

We next show that John domains satisfy the interior corkscrew condition. (Note that the only unbounded John domain is X itself which is excluded from our considerations.)

Lemma 8.9. *Let U be a bounded c_J -John domain with $0 < c_J \leq 1$. Then U satisfies the interior corkscrew condition with parameters $\kappa = c_J^2/4$, 0 and $\text{diam}(U)$.*

Proof. Let x_U be the John center of U . For any $x \in U$ we find a c_J -John curve γ connecting x and x_U , i.e., (8.2) holds. In particular, $\delta_U(x_U) \geq c_J \ell(\gamma) \geq c_J d(x, x_U)$. Taking the supremum with respect to $x \in U$, we obtain $\delta_U(x_U) \geq c_J \text{diam}(U)/2$.

Now let $x \in U$ and $0 < r < \text{diam}(U)$. We claim that $U \cap B(x, r)$ contains a ball of radius $c_J^2 r/4$. If $\delta_U(x) \geq c_J^2 r/4$, then $U \cap B(x, r) \supset B(x, c_J^2 r/4)$. So, suppose that $\delta_U(x) < c_J^2 r/4$. Then

$$\frac{c_J r}{2} < \frac{c_J \text{diam}(U)}{2} \leq \delta_U(x_U) \leq \delta_U(x) + d(x, x_U) < \frac{c_J^2 r}{4} + d(x, x_U),$$

so that $d(x, x_U) > c_J r/4$. Let γ be a John curve connecting x and x_U . We find a point $y \in \gamma$ with $d(x, y) = c_J r/4$. Then $\delta_U(y) \geq c_J \ell(\gamma(x, y)) \geq c_J^2 r/4$. Hence,

$$B(y, c_J^2 r/4) \subset U \cap B(x, c_J^2 r/4 + c_J r/4) \subset U \cap B(x, r),$$

as required. \square

Theorem 8.10. *Let $\mathcal{J}(c_J)$ be the family of all bounded c_J -John domains. If $\mathcal{J}(c_J)$ satisfies Definition 8.1 (i), in particular if $0 < c_J \leq 1/L$ (where L is the quasiconvexity constant), then it is capacitarily stable.*

Proof. This follows directly from Lemma 8.9 and Theorem 8.5, together with the following remark. \square

Remark 8.11. It is easy to see that if X is a geodesic space, then $B(x, r)$ is a 1-John domain with John center x . On the other hand, the counterexample in Theorem 1.2 is not geodesic with respect to d and an ordinary ball $B(x, r)$ need not be a John domain.

Since X is always geodesic with respect to d_{in} , it follows that $B_{\text{in}}(x, r)$ is a 1-John domain with respect to d_{in} , and hence a $(1/L)$ -John domain with respect to d . This, in particular, means that for $c_J \leq 1/L$ the family $\mathcal{J}(c_J)$ is nonempty. Furthermore, as X is quasiconvex, the family $\mathcal{J}(c_J)$ satisfies Definition 8.1 (i).

The following example shows that there are unbounded metric spaces with no c_J -John domains of large diameter, when c_J is close to 1. Thus, $\mathcal{J}(c_J)$ is not always capacitarly stable since Definition 8.1 (i) is violated in such situations.

Example 8.12. Consider

$$X = \{(x, y) \in \mathbb{R}^2 : |y - \cos x| \leq \tfrac{1}{2}\},$$

equipped with the Euclidean metric and the 2-dimensional Lebesgue measure, see Figure 2. Since X is biLipschitz equivalent to $[0, 1] \times \mathbb{R}$, it follows that the measure on X is doubling and supports a 1-Poincaré inequality.

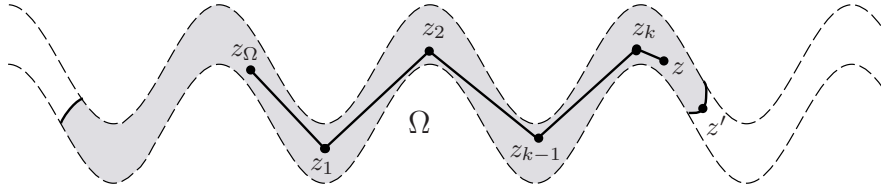


FIGURE 2. No c_J -John domains of large diameter if $c_J > \pi/\sqrt{1+\pi^2}$.

Let $\Omega \subset X$ be a bounded domain with $\text{diam}(\Omega) > 2\pi + 3$ and let $z_\Omega = (x_0, y_0) \in \Omega$ act as a John center. By translation of Ω , we can assume that $|x_0| \leq \pi$. Find $z' = (x', y') \in \partial\Omega$ so that $|x'|$ is as large as possible. Because of the symmetry of X and as $\text{diam}(\Omega) > 2\pi + 3$, we can assume that $x' \in (k\pi, (k+1)\pi]$ for some integer $k \geq 1$ since $\text{diam}([- \pi, \pi] \times [-\frac{3}{2}, \frac{3}{2}]) < 2\pi + 3$.

A simple geometric argument then shows that $\delta_\Omega(z_\Omega) \leq |z_\Omega| + |z'| \leq (k+2)\pi + 3$ and that any curve γ in Ω , which connects z_Ω with a point $z = (x, y) \in \Omega$, where $x > k\pi$, intersects vertical lines of x -coordinate $j\pi$, and hence contains points $z_j = (j\pi, y_j)$ with $|y_j - \cos j\pi| \leq \frac{1}{2}$ for $j = 1, \dots, k$. Since $|y_j - y_{j+1}| \geq 1$, we conclude that

$$\ell(\gamma) \geq \sum_{j=1}^{k-1} |z_j - z_{j+1}| \geq (k-1)\sqrt{1+\pi^2}.$$

Since

$$\frac{\delta_\Omega(z_\Omega)}{\ell(\gamma)} \leq \frac{(k+2)\pi + 3}{(k-1)\sqrt{1+\pi^2}} \rightarrow \frac{\pi}{\sqrt{1+\pi^2}} < 1, \quad \text{as } k \rightarrow \infty,$$

we see that for every $c_J > \pi/\sqrt{1+\pi^2}$, there exists $r > 0$ such that there are no c_J -John domains in X with diameter at least r .

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